

Precanonical perspective in quantum gravity*

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Quantization of general relativity in metric variables using “precanonical” quantization based on the De Donder-Weyl covariant Hamiltonian formulation is outlined. Elements of classical geometry needed to formulate the wave equation emerge from a self-consistency with the underlying quantum dynamics of the metric in this sense ensuring the background independence of the formulation.

1. INTRODUCTION

In spite of the noticeable progress in the quantum theory of gravity during the last decade, mainly owing to the Ashtekar program of non-perturbative canonical quantum gravity and the string/M-theory, it is difficult to escape a feeling that the genuine conceptual foundations of the synthesis of general relativity and the quantum theory are still awaiting of their discovery. In particular, the drastic differences between the way the physics is described general relativistically and quantum theoretically urge us to inquire if the presently known procedures of field quantization, that is the specific way the quantum paradigm is implemented, are adequate to the problem of quantization of gravity. The particular concern is due to the different status of the time variable in quantum theory and in general relativity, in addition to the characteristic to the latter diffeomorphism covariance and a dynamical character of the space-time. The main objective of what we call the precanonical approach to field quantization is to elaborate a procedure which would treat all space-time variables on equal footing, more in accordance with the relativity theory.

The idea of precanonical approach is suggested

by a long-known in the calculus of variations fact that the Hamiltonian formulation can be alternatively extended to field theory in the form of the De Donder-Weyl (DW) canonical equations [1]

$$\partial_\mu y^a = \frac{\partial H}{\partial p_a^\mu}, \quad \partial_\mu p_a^\mu = -\frac{\partial H}{\partial y^a}, \quad (1)$$

where given a Lagrangian density $L(y^a, y_\mu^a, x^\nu)$, a function of field variables y^a , their space-time derivatives (first jets) y_μ^a and space-time variables x^μ , one introduces new Hamiltonian-like variables $p_a^\mu := \partial L / \partial y_\mu^a$ (*polymomenta*) and $H = H(y^a, p_a^\mu, x^\nu) := \partial_\mu y^a p_a^\mu - L$ (*the DW Hamiltonian function*). Obviously, this formulation is manifestly covariant, it treats space and time variables on equal footing, i.e. requires no usual $3 + 1$ decomposition, and corresponds to what could be viewed as a “multi-time” generalization of the Hamiltonian formulation from mechanics to field theory. No picture of fields as infinite-dimensional mechanical systems evolving in time is implied in (1); instead, fields are described rather as systems varying in space and time. These features make formulation (1) an attractive alternative to the conventional “instantaneous” Hamiltonian formalism as a basis of quantization, especially in the context of general relativity. Besides, the obstacles to the DW Legendre transformation $y_\mu^a \rightarrow p_a^\mu$ in general are different from the usual constraints, thus suggesting a possibility of surmounting the usual constraints analysis when quantizing.

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The term “precanonical” refers to the fact that formulation (1) and the related constructions are in a sense intermediate between the covariant Lagrangian description and the “instantaneous” canonical Hamiltonian description; in mechanics precanonical structures coincide with the canonical ones while in field theory they are different.

Field quantization stemming from formulation (1) has been considered in [2–4]; its application to general relativity has been discussed in a recent preprint by the author [5] and will be outlined below. Briefly, quantization based on DW Hamiltonian formulation (1) leads to a generalization of quantum theoretic formalism in which the space-time Clifford algebra replaces the algebra of complex numbers (=the Clifford algebra of $(0+1)$ -dimensional space-time!) in quantum mechanics. The Clifford algebra appears when quantizing the Poisson brackets which in DW theory are defined on differential forms [6,7] (c. f. [8]). This results in representing polymomenta by operators

$$\hat{p}_a^\mu = -i\hbar\kappa\gamma^\mu \frac{\partial}{\partial y^a}, \quad (2)$$

which act on spinor (or the Clifford algebra valued) wave functions $\Psi = \Psi(y^a, x^\mu)$; γ^μ 's denote the imaginary units of the space-time Clifford algebra; the constant κ of the dimension $[\text{length}]^{-3}$ ensures the dimensional consistency of (2) and is interpreted as a quantity of the ultra-violet cut-off or the fundamental length scale [2,4]. The wave function is supposed to fulfill a generalized Schrödinger equation

$$i\hbar\kappa\gamma^\mu\partial_\mu\Psi = \hat{H}\Psi, \quad (3)$$

where \hat{H} is the operator form of the DW Hamiltonian function. This equation was found to be consistent with several aspects of the correspondence principle [3,4], for example, it leads to an analogue of the Ehrenfest theorem and can be reduced to the field theoretic DW Hamilton-Jacobi equation (with some additional conditions) in the classical limit. Unfortunately, the details of the relationship between the standard quantum field theory and the present formulation so far remain poorly understood. A possible connection with the Schrödinger functional picture has been discussed in [2,4].

2. PRECANONICAL QUANTUM GENERAL RELATIVITY: AN OUTLINE

When applying the above approach to gravity the configuration space is to be a bundle of symmetric second rank tensors over the space-time and the wave function is to be a function on this space: $\Psi = \Psi(g^{\mu\nu}, x^\mu)$. The Schrödinger equation for this wave function is obtained by first writing a curved space-time version of (3) and then replacing the metric and the connection by the corresponding operators. This leads to the following guess concerning the wave equation for quantum gravity within the precanonical approach:

$$i\hbar\kappa e\widehat{\nabla}\Psi = \hat{\mathcal{H}}\Psi, \quad (4)$$

where $\hat{\mathcal{H}}$ is the operator form of DW Hamiltonian density of gravity, $\mathcal{H} := \sqrt{g}H$, where $\sqrt{g} := \sqrt{|\det(g_{\mu\nu})|} =: e$, and $\widehat{\nabla} := \gamma^\mu(\partial_\mu + \hat{\theta}_\mu)$ denotes the quantized covariant Dirac operator in which γ -matrices are such that $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ and $\hat{\theta}_\mu$ is the spinor connection operator. Recall that classically $\theta_\mu = \theta^{\alpha\beta}{}_\mu\gamma_{[\alpha}\gamma_{\beta]}$, $\gamma^\mu = e_A^\mu\gamma^A$ and

$$\theta^{\alpha\beta}{}_\mu = g^{\nu[\beta}(\Gamma_{\mu\nu}^\alpha - e_A^\alpha\partial_\mu e_\nu^A). \quad (5)$$

Since the quantum gravity possesses an intrinsic fundamental length scale, the Planck length ℓ , one can expect that $\kappa \sim \ell^{-3}$.

To find an operator realization of the quantities involved in (4) we first have to formulate the Einstein equations in DW Hamiltonian form. It is given by the set of equations [9]

$$\partial_\alpha h^{\beta\gamma} = \partial\mathcal{H}/\partial Q_{\beta\gamma}^\alpha, \quad \partial_\alpha Q_{\beta\gamma}^\alpha = -\partial\mathcal{H}/\partial h^{\beta\gamma}, \quad (6)$$

where $h^{\alpha\beta} := \sqrt{g}g^{\alpha\beta}$ are field variables,

$$Q_{\beta\gamma}^\alpha := \frac{1}{8\pi G} \left(\delta_{(\beta}^\alpha \Gamma_{\gamma)\delta}^\delta - \Gamma_{\beta\gamma}^\alpha \right) \quad (7)$$

are corresponding polymomenta and

$$\mathcal{H} := 8\pi G h^{\alpha\gamma} \left(Q_{\alpha\beta}^\delta Q_{\gamma\delta}^\beta - \frac{1}{3} Q_{\alpha\beta}^\beta Q_{\gamma\delta}^\delta \right) \quad (8)$$

is the DW Hamiltonian function of gravity. The above formulation of the Einstein equations has a deeper foundation in the theory of Lepagean equivalents in the calculus of variations [9].

Now, polymomenta can be quantized according to the rule (2) adapted to curved space-time

$$\hat{Q}_{\beta\gamma}^\alpha = -i\hbar\kappa\gamma^\alpha \left\{ \sqrt{g} \frac{\partial}{\partial h^{\beta\gamma}} \right\}_{ord}, \quad (9)$$

where the notation $\{\dots\}_{ord}$ refers to the ordering ambiguity of the expression inside the curly brackets. Plugging (9) into (8) we obtain

$$\hat{\mathcal{H}} = -\frac{16\pi}{3}G\hbar^2\kappa^2 \left\{ \sqrt{g}h^{\alpha\gamma}h^{\beta\delta} \frac{\partial}{\partial h^{\alpha\beta}} \frac{\partial}{\partial h^{\gamma\delta}} \right\}_{ord} \quad (10)$$

When formulating the left hand side of eq. (4) we are led to the fundamental difficulties related to the fact that (i) conceptually, the Dirac operator generally refers to a classical space-time background which is ought to be avoided in quantum gravity and (ii) technically, the last term in the spinor connection (5) cannot be expressed in terms of metric variables. We deal with these difficulties by observing that the tetrads do not enter the present DW formulation of General Relativity underlying the quantization and, therefore, can be treated only as non-quantized x -dependent quantities $\tilde{e}_A^\mu(x)$. The correspondence principle then implies that they should be related to the mean value of the metric as follows

$$\tilde{e}_A^\mu(x)\tilde{e}_B^\nu(x)\eta^{AB} = \langle g^{\mu\nu} \rangle(x) \quad (11)$$

where

$$\langle g^{\mu\nu} \rangle(x) = \int [dg^{\alpha\beta}] \bar{\Psi}(g, x) g^{\mu\nu} \Psi(g, x), \quad (12)$$

and

$$[dg^{\alpha\beta}] = g^{5/2} \prod_{\alpha \leq \beta} dg^{\alpha\beta} \quad (13)$$

is the invariant integration measure on the 10-dimensional space of metric components (c.f. [10]). Note that (12) is well-defined mathematically as a smooth field.

To quantize the connection coefficients let us note that classically (c.f. (7))

$$\Gamma_{\beta\gamma}^\alpha = 8\pi G \left(\frac{2}{3} \delta_{(\beta}^\alpha Q_{\gamma)\delta}^\delta - Q_{\beta\gamma}^\alpha \right). \quad (14)$$

Now, using (5) and (9) we can write

$$\hat{\theta}^{\alpha\beta}{}_\mu = -8\pi i G \hbar \kappa \left\{ h^{\nu[\beta} \left(\frac{2}{3} \delta_{(\mu}^{\alpha]} \gamma^\sigma \frac{\partial}{\partial h^{\nu]\sigma}} \right. \right.$$

$$\left. \left. - \gamma^{\alpha]} \frac{\partial}{\partial h^{\mu\nu}} \right) \right\}_{ord} + \tilde{\theta}^{\alpha\beta}{}_\mu(x). \quad (15)$$

This expression involves the ordering dependent operator part $(\theta^{\alpha\beta}{}_\mu)^{op}$ and an auxiliary spinor connection part $\tilde{\theta}^{\alpha\beta}{}_\mu(x)$ which (i) accounts for the term in (5) which cannot be expressed in metric variables (hence, cannot be quantized) and (ii) ensures the transformation law of $\hat{\theta}^{\alpha\beta}{}_\mu$ is that of a spinor connection. Our assumption is that $\tilde{\theta}^{\alpha\beta}{}_\mu(x)$ is given by the standard formula

$$\tilde{\theta}^{\alpha\beta}{}_\mu(x) = 2g^{\gamma[\alpha} \tilde{e}_B^{\beta]} \partial_{[\mu} \tilde{e}_{\gamma]}^B + g^{\alpha\gamma} g^{\delta\beta} \tilde{e}_\mu^B \partial_{[\delta} \tilde{e}_{\gamma]B} \quad (16)$$

where the tetrad field $\tilde{e}_A^\mu(x)$ is given by (11).

Now, we can formulate the diffeomorphism covariant wave equation for quantum gravity:

$$i\hbar\kappa\tilde{\nabla}\Psi + i\hbar\kappa(\sqrt{g}\gamma^\mu\theta_\mu)^{op}\Psi = \hat{\mathcal{H}}\Psi, \quad (17)$$

where $\tilde{\nabla} = \tilde{e}_A^\mu(x)\gamma^A(\partial_\mu + \tilde{\theta}_\mu(x))$ is the Dirac operator constructed using the self-consistent field $\tilde{e}_A^\mu(x)$ and

$$(\sqrt{g}\gamma^\mu\theta_\mu)^{op} = -4\pi i G \hbar \kappa \left\{ \sqrt{g}h^{\mu\nu} \frac{\partial}{\partial h^{\mu\nu}} \right\}_{ord} \quad (18)$$

is the term corresponding to the operator part of the spinor connection. The idea behind eq. (17) is that classical geometric structures needed to formulate the wave equation are introduced as approximate averaged notions in a self-consistent with the underlying quantum dynamics (determined by $\hat{\mathcal{H}}$) way. As a consequence of condition (11) eq. (17) essentially becomes a non-linear integro-differential equation describing the non-trivial way in which the wave function Ψ specifies, or “lays down”, the space-time geometry it propagates on.

To complete the description, we also need to impose a gauge-type condition in order to distinguish the physically relevant information. For example, the De Donder-Fock harmonic gauge can be imposed in the form

$$\partial_\mu \langle \sqrt{g}g^{\mu\nu} \rangle(x) = 0. \quad (19)$$

In the present context this is a gauge condition on the wave function $\Psi(g^{\mu\nu}, x^\nu)$ rather than on the metric field.

3. CONCLUSION

The De Donder-Weyl Hamiltonian formulation of the field equations leads to the procedure of quantization of fields, which we suggested to call precanonical, treating space and time variables on equal footing. When applied to general relativity in metric variables this framework leads to a Dirac-like wave equation (17), non-linear and integro-differential, with self-consistently incorporated classical geometric structures. No *arbitrarily* fixed background structure is present in the formulation; background independence is ensured by self-consistency which in its turn is dictated by the correspondence principle. The averaged self-consistent space-time serves its usual role: to order events (here, different possible configurations of the wave function in the metric space where its *linear* quantum dynamics is given by DW Hamiltonian operator $\hat{\mathcal{H}}$) and to interpolate between them. This is the only way to describe physics we are sure about at the present: to describe it in space and time. Here we do have a quantum dynamics of the wave function in the metric space but we also do need classical space-time to order and join together the configurations of the wave function in the metric space (the fibre) in different points of the space-time (the base). Our wave equation prescribes how this ordering is achieved and how, as a consequence of this, the space-time is gaining its metric structure. This reference to classical space-time, even though self-consistent, may well be an approximation. “Quantum space-time”, usually thought to be an essential ingredient of quantum gravity, would imply a totally different way of describing physics. Our approach suggests that this could be achieved by attributing a proper sense to the “operator of the Dirac operator” in the left hand side of (4) without referring to classical geometric notions. A. Connes’ non-commutative geometry [11] can be mentioned as an example of a mathematical framework achieving this goal.

Potential advantages of the present approach are (i) the manifest covariance of the foundations and the results and (ii) the luck of serious mathematical problems with underlying mathematical constructions (as opposite to, e.g., the Wheeler-

De Witt geometrodynamics). It also offers a framework for discussing the problem of emergence of classical space-time in quantum gravity and has a potential to enlighten the problem of interpretation of quantum formalism in quantum cosmology: the quantum system described by (17) is “self-referential” in the sense that the classical self-consistent tetrad field can be viewed as a model of the observing degrees of freedom explicitly entering into the description of quantum dynamics. We hope that these intriguing features are a sufficient justification of the further analysis and development of the precanonical approach to quantum fields and quantum gravity, in spite of its so far unclarified connections to the standard quantum field theory.

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